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On angles of separation in Stokes flows

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A two-dimensional Stokes flow close to the line of contact of two touching cylinders or three-dimensional axisymmetric Stokes flow close to the point of contact of two touching bodies is shown in general to separate into infinite sets of eddies with angles of separation from the bodies which tend to 58.61° as the line or point of contact is approached. The flow near the vertex of a conical cusp is shown to be a system of nested toroidal vortices and the separation angles tend to 45.25° as the vertex is approached. Stokes flow between parallel planes or within a circular cylinder is shown in general to separate far from the generating disturbances with cellular eddy structure and separation angles which tend to 58.61° and 45.25° respectively. The mathematical equivalence of the various problems is established.

1. Introduction

In the past few years examples of Stokes flows that exhibit separation and eddy formation have appeared in the literature. Moffatt (1964) showed that in general the flow between planes which intersect at an angle of less than about 146.3° has an eddy structure close to the corner and an infinite sequence of line vortices is formed whose strength diminishes exponentially as the corner is approached. More recent investigations have shown that the phenomenon of multiple eddy formation is widespread in both two- and three-dimensional flows involving either planes or finite-sized bodies, and much of this work is reviewed in the articles by Hasimoto & Sano (1980) and Liron & Blake (1981).

Davis *et al.* (1976) showed that, in an axisymmetric streaming flow past two equal spheres in contact, the flow separates from the spheres near the point of contact and an infinite set of nested toroidal vortices is formed. For two-dimensional linear shear or stagnation-point flow over a cylinder touching a plane, Davis & O'Neill (1977*a*, *b*) showed that an infinite set of line vortices is formed near the line of contact. An interesting feature arising from these three studies is that the angle at which the separating streamlines detach from (or attach to) any of the boundaries tends to the same limit 58.61° as the point or line of contact is approached. This leads one to conjecture if this is a universal angle for separation from bodies of arbitrary shape in either two- or three-dimensional Stokes flows, and in this paper it is shown that this is indeed generally true for both two-dimensional flows about cylindrical bodies in contact and for three-dimensional axisymmetric flows about axially symmetric bodies in contact.

Another type of separating Stokes flow is the axisymmetric streaming flow past a closed torus. In this flow, Dorrepaal *et al.* (1976) showed that infinite sets of nested ring vortices form within the central cylindrical cusps. Bourot (1975) showed, in his numerical study of axisymmetric streaming past a cardioid of revolution, that a similar eddy structure develops in the cusp. In this paper it is shown that this is a feature of any axisymmetric flow about an axisymmetric body of arbitrary shape with cylindrical cusps. The angle of separation of the streamlines from the body now tends to the constant value 45.25° as the vertex of the cusp is approached. These examples illustrate how a local geometrical feature can profoundly influence the character of the local flow to such an extent that a purely local analysis can provide the form of the stream function, apart from a scaling factor, and hence all local flow properties can be qualitatively deduced. The elliptic nature of the boundary-value problem for the stream function must of course necessitate reference to the exact form of the global flow about the body in order to determine the value of the scaling factor as pointed out by Michael & O'Neill (1977) in the context of separation from quasiplanar boundaries.

Moffatt's analysis of corner flow between intersecting planes included an example of cellular eddy flow between parallel planes which can be interpreted as the limiting form of the corner flow as the angle between the planes approaches zero in such a way that their distance apart approaches a constant value. Hancock (1983) has calculated the angle of separation for flow in corners of various angles and has shown that, in this limit, antisymmetric flow separates from the parallel plane walls at 58.61°. In this paper, it is shown that, in general, any Stokes flow between parallel plane walls whose velocity decays to zero in at least one direction, say $x \to +\infty$, ultimately has an antisymmetric cellular eddy structure with separation streamlines detaching from the walls at 58.61°. For the corresponding flow in a circular cylinder a similar effect occurs, with the flow ultimately possessing a cellular toroidal eddy structure with the separating streamlines now detaching at 45.25° from the cylinder. The fact that these separation angles are identical with those occurring as a result of local boundary geometry is shown to follow, since by a suitable inversion transformation the asymptotic solution associated with a local boundary geometrical property of the type considered, i.e. contact or conical cusp, is mathematically equivalent to the far-field asymptotic solution for flow between parallel planes or within a circular cylinder.

2. Flow near the contact line of two touching cylindrical bodies

Let us consider a two-dimensional Stokes flow near the line of contact of two cylindrical bodies B_1 and B_2 at rest and let O be the point where any plane section perpendicular to the axes of the bodies intersects the line of contact. Let the boundary sections of the bodies have circles of curvature C_1 and C_2 at O and the radii of curvature be R_1 and R_2 respectively. The geometry is illustrated in figure 1.

If (x, y) denote Cartesian coordinates in this plane with O as origin, the equation of B_1 can be written as

$$y = F_1(x) = F_1(0) + F_1'(0)\,x + \tfrac{1}{2}F_1''(0)\,x^2 + \dots \tag{2.1}$$

in the neighbourhood of O. But $F_1(0) = F_1'(0) = 0$ and $R_1 = \{1 + |F_1'(0)|^2\}^{\frac{3}{2}}/|F_1''(0)|$ assuming $F_1''(0) \neq 0$. Consequently $F''(0) = \pm R_1^{-1}$, with the + sign chosen if B_1 is concave towards the positive y-axis at O. Without loss of generality we may assume this configuration and the equation of B_1 is accordingly given locally by the equation

$$y = \frac{x^2}{2R_1} + O(x^3). \tag{2.2}$$

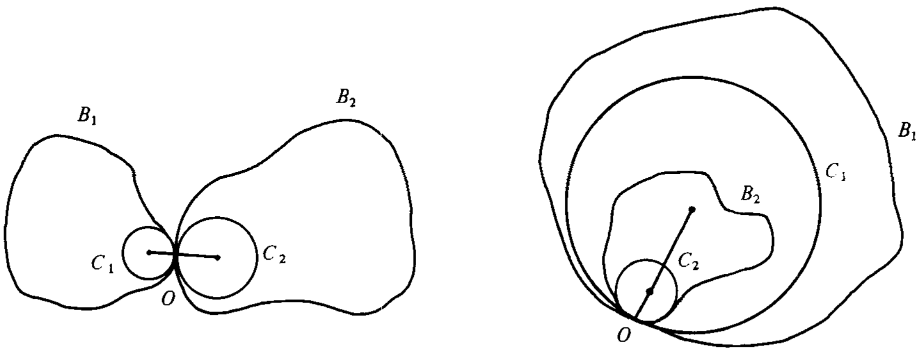

FIGURE 1. Two touching cylindrical bodies.

The equation of the circle of curvature C_1 is $(y-R_1)^2+x^2 = R_1^2$, giving

$$y = \frac{x^2}{2R_1}+O(x^4) \tag{2.3}$$

near O, showing that B_1 and C_1 have the same equation sufficiently near to the point of contact O provided that $F_1'(0) \neq 0$. Similarly the body B_2 and its circle of curvature C_2 have the same equation near O, and this is given by

$$y = \mp\frac{x^2}{2R_2}+O(x^3), \tag{2.4}$$

where the $-$ or $+$ sign is chosen according as B_1 and B_2 make contact externally or internally, as illustrated in figure 1. On making x and y dimensionless relative to R_1 and writing $|\alpha| = R_1/R_2$, with the sign of α negative if B_1 and B_2 touch externally, the equations of B_1 and B_2 are given locally by

$$y = \tfrac{1}{2}x^2, \quad y = \tfrac{1}{2}\alpha x^2 \tag{2.5}$$

respectively, showing that $2y/x^2$ is a similarity variable for the problem. This suggests introducing variables ξ, η defined by

$$\xi = \frac{2y}{x^2}, \quad \eta = \frac{2}{x}. \tag{2.6}$$

The curves $\xi =$ constant are parabolas which touch at $x = 0$, and the curves $\eta =$ constant are straight lines. The bodies B_1 and B_2 are given by $\xi = 1$ and $\xi = \alpha$ respectively, as illustrated in figure 2. The origin $x = y = 0$ corresponds to $\eta = \infty$.

The level surfaces in this coordinate system are generally not orthogonal but approach orthogonality as the point of contact is approached, corresponding to $\eta \to \infty$. It is assumed that the flow in the neighbourhood of the point of contact O is a steady or quasi-steady Stokes flow, so the boundary-value problem for the stream function ψ is to find a solution of

$$\nabla^4\psi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = 0 \tag{2.7}$$

satisfying $\psi = \partial\psi/\partial n = 0$ on B_1 and B_2, with n denoting distance along the normal to either boundary. To determine the complete solution to the problem, an appropriate condition defining the generation of the global flow away from O must be prescribed, but this condition is not required in order to determine the structure of the flow in

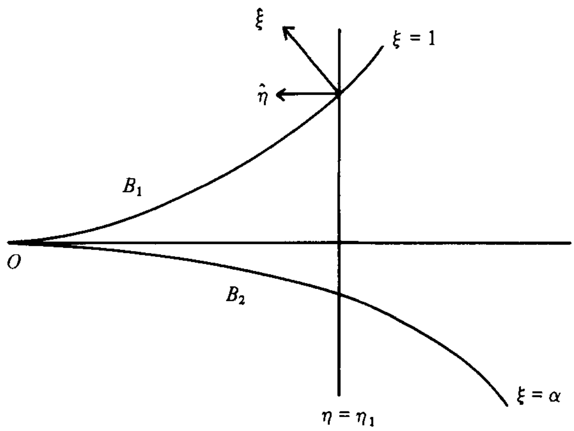

FIGURE 2. The level surfaces ξ = constant and η = constant.

the neighbourhood of O. In terms of the ξ, η variables, the boundary conditions are

$$\psi = \frac{\partial \psi}{\partial \xi} = 0 \quad (\xi = 1, \alpha), \tag{2.8}$$

and noting that if the Laplace operator is expressed in these coordinates,

$$\nabla^2 f(\xi, \eta) \sim \frac{\eta^4}{4}\left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right) \tag{2.9}$$

as $\eta \to \infty$, it follows that for large η the solution for ψ in separated variables takes the form

$$\psi = \eta^{-2}\{(A+\xi C)\sinh s\xi + (B+\xi D)\cosh s\xi\} \mathrm{e}^{\pm i s \eta}, \tag{2.10}$$

where the parameter s may be complex and A, B, C, D are functions of s. The + or − sign is chosen so that $\psi \to 0$ as $\eta \to \infty$ and it is understood that the real part of the right-hand side of (2.10) is taken. The boundary conditions on the bodies require that

$$(A+\xi C)\sinh s\xi + (B+\xi D)\cosh s\xi = 0 \quad (\xi = 1, \alpha), \tag{2.11}$$

$$(sA+\xi sC+D)\cosh s\xi + (sB+\xi sD+C)\sinh s\xi = 0 \quad (\xi = 1, \alpha), \tag{2.12}$$

giving four equations for A, B, C, D. Consistency of these homogeneous equations requires that

$$s^2(1-\alpha)^2 - \sinh^2 s(1-\alpha) = 0, \tag{2.13}$$

and the solutions of this equation give rise to the set of permissible eigenvalues for the problem. On writing $s(1-\alpha) = iz$, (2.13) gives

$$\sin z + z = 0 \quad \text{or} \quad \sin z - z = 0. \tag{2.14}$$

Equations (2.14) are the same pair of equations which arise in the analyses of Dean & Montagnon (1949) and Moffatt (1964) for flow in a corner formed by intersecting planes. Apart from $z = 0$, which leads to the trivial solution for ψ, the roots of these equations are respectively $z = \pm\lambda_n$, $z = \pm\mu_n$, arranged in the order of increasing real parts, and their conjugates. These roots have been accurately tabulated by Buchwald (1964) and the first few are listed below

$$\left.\begin{aligned} \lambda_1 &= 4.213+2.251i, \quad \lambda_2 = 10.713+3.103i, \\ \mu_1 &= 7.499+2.769i, \quad \mu_2 = 13.900+3.352i. \end{aligned}\right\} \tag{2.15}$$

It therefore follows that, near the line of contact, the stream function ψ generally has the asymptotic form

$$\psi = \eta^{-2}\,\mathrm{Re}\,A_1 f_1(\xi)\,\mathrm{e}^{-\lambda_1\eta/(1-\alpha)}, \tag{2.16}$$

unless $A_1 = 0$, in which case the appropriate asymptotic form is then

$$\psi = \eta^{-2}\,\mathrm{Re}\,\hat{A}_1 g_1(\xi)\,\mathrm{e}^{-\mu_1\eta/(1-\alpha)}, \tag{2.17}$$

where

$$f_1(\xi) = (1-\xi)\sin\frac{\lambda_1(\xi-\alpha)}{1-\alpha} + (\xi-\alpha)\sin\frac{\lambda_1(1-\xi)}{1-\alpha}, \tag{2.18}$$

$$g_1(\xi) = (1-\xi)\sin\frac{\mu_1(\xi-\alpha)}{1-\alpha} - (\xi-\alpha)\sin\frac{\mu_1(1-\xi)}{1-\alpha}. \tag{2.19}$$

The functions $f_1(\xi)$ and $g_1(\xi)$ are respectively even and odd functions about $\xi = \frac{1}{2}(\alpha+1)$. Thus, in general, any Stokes flow of the type considered is antisymmetric about the midsurface $\xi = \frac{1}{2}(\alpha+1)$ sufficiently close to the line of contact, and its stream function is given asymptotically by (2.16).

Although the constant A_1 is indeterminate without reference to the global flow about the bodies, many qualitative features of the flow behaviour can be obtained by examining (2.16). For instance it is clear that, for any fixed value of ξ, the stream function vanishes an infinity of times as $\eta \to \infty$. This means that there are an infinity of branches of the streamline $\psi = 0$ linking the bodies in the neighbourhood of the line of contact, indicating the existence of an infinite set of nested eddies similar in type to those shown to exist by Davis & O'Neill (1977*a*, *b*) for a cylinder touching a plane wall in a linear shear flow. The equations of the eddy cell boundaries where $\psi = 0$ are

$$\eta(1-\alpha)^{-1}\,\mathrm{Im}\,(\lambda_1) = \arg A_1 + \arg f_1(\xi) + (m-\tfrac{1}{2})\,\pi \quad (\xi \neq 1, \alpha), \tag{2.20}$$

showing that the difference between values of η on successive branches is the constant $\pi(1-\alpha)/\mathrm{Im}\,(\lambda_1)$. Where these curves intersect the bodies are the points of separation or attachment for the flow. The points where separation occurs on B_1 are given by $\xi = 1, \eta = \eta_s$, where

$$\frac{\partial^2\psi}{\partial\xi^2} = 0 \quad (\xi = 1,\ \eta = \eta_s),$$

whose solutions are given by

$$\eta_s(1-\alpha)^{-1}\,\mathrm{Im}\,(\lambda_1) = \arg A_1 + \arg\lambda_1 + \arg(1+\cos\lambda_1) + (m-\tfrac{1}{2})\,\pi. \tag{2.21}$$

The values of η at the points of separation on body B_2 are also given by (2.21). On expanding ψ in the neighbourhood of a typical separation point, the angle of separation can be found. For body B_1

$$\psi(\xi,\eta) \simeq \tfrac{1}{2}(1-\xi)^2(\eta-\eta_s)\frac{\partial^3\psi(1,\eta_s)}{\partial\xi^2\,\partial\eta} - \tfrac{1}{6}(1-\xi)^3\frac{\partial^3\psi(1,\eta_s)}{\partial\xi^3}.$$

The angle γ at which the streamline separates from this surface is accordingly

$$\gamma = \tan^{-1}\left[3\frac{\partial^3\psi}{\partial\xi^2\,\partial\eta}\Big/\frac{\partial^3\psi}{\partial\xi^3}\right] \quad (\xi = 1,\ \eta = \eta_s). \tag{2.22}$$

However, at this separation point

$$\frac{\partial^3\psi}{\partial\xi^2\,\partial\eta} = 2\eta^{-2}(1-\alpha)^{-1}\,\mathrm{Re}\,\{A_1\lambda_1^2(1+\cos\lambda_1)\,\mathrm{e}^{-\lambda_1\eta/(1-\alpha)}\},$$

$$\frac{\partial^3\psi}{\partial\xi^3} = -2\eta^{-2}(1-\alpha)^{-2}\,\mathrm{Re}\,\{A_1\lambda_1^3\,\mathrm{e}^{-\lambda_1\eta/(1-\alpha)}\},$$

and noting that at the separation point

$$\frac{\partial^2\psi}{\partial\xi^2} = -2\eta^{-2}(1-\alpha)^{-1}\,\mathrm{Re}\{A_1\lambda_1(1+\cos\lambda_1)\,\mathrm{e}^{-\lambda_1\eta(1-\alpha)}\} = 0,$$

it can be shown that (2.22) reduces to

$$\gamma = \tan^{-1}\frac{3\,\mathrm{Im}\,(-\lambda_1)}{\mathrm{Im}\,\{\lambda_1^2/(1+\cos\lambda_1)\}}$$

$$= \tan^{-1}\frac{3\,\mathrm{Im}\,(\lambda_1)}{\mathrm{Im}\,(\cos\lambda_1)} = 58.61°. \tag{2.23}$$

A similar analysis reveals that this is also the value of the angle of separation from the body B_2 as the line of contact is approached. This is the angle of separation which was shown by Davis & O'Neill (1977*a*, *b*) to occur in two-dimensional linear shear or stagnation-point flow over a circular cylinder touching a plane as the line of contact is approached.

The angle of separation is evidently 58.61°, provided that the dominant term in the stream function is given by (2.16), and this will generally be the case for an arbitrary flow when ψ is neither odd nor even about the midsurface $\xi = \frac{1}{2}(\alpha+1)$. If, however, the flow is purely symmetric about this surface, as in the streaming flow past two cylinders in contact when the direction of the stream is perpendicular to the plane containing the axes of the cylinders, which was studied by Dorrepaal & O'Neill (1978), then only the terms involving the eigenvalues μ_n appear in the solution, and the appropriate asymptotic form for ψ is then given by (2.17). The corresponding angle of separation from each of the bodies is then

$$\tan^{-1}\left[\frac{-3\,\mathrm{Im}\,(\mu_1)}{\mathrm{Im}\,(\cos\mu_1)}\right] = 48.15°. \tag{2.24}$$

This was the angle obtained by Davis (1979) for the flow considered by Dorrepaal & O'Neill. However, for unequal-sized cylinders the purely symmetrical property of the flow is lost and the limiting separation angle is then 58.61°. If the stream flows parallel to the plane containing the axes of the cylinders, the flow is purely antisymmetrical for equal-sized cylinders, thus, for any two cylinders in this type of stream, the separation angle accordingly has the limiting value of 58.61°.

3. Flow near the point of contact of two touching axisymmetric bodies

Two axisymmetric bodies B_1 and B_2 touch at O and have a common axis of symmetry passing through O. Obvious examples are two spheres or a spheroid touching a plane, and the geometry is illustrated in figure 3.

If the principal radii of curvature of the bodies are respectively R_1 and R_2, and (r,θ,z) denote cylindrical polar coordinates with pole at O and the z-axis coinciding with the axis of symmetry of the bodies, it follows that in the neighbourhood of O the equations of B_1 and B_2 are given by

$$z = \frac{r^2}{2R_1} + O(r^3), \tag{3.1}$$

$$z = \frac{\pm r^2}{2R_2} + O(r^3) \tag{3.2}$$

respectively. On making r, z dimensionless relative to R_1, and writing $R_2 = |\alpha|^{-1}R_1$,

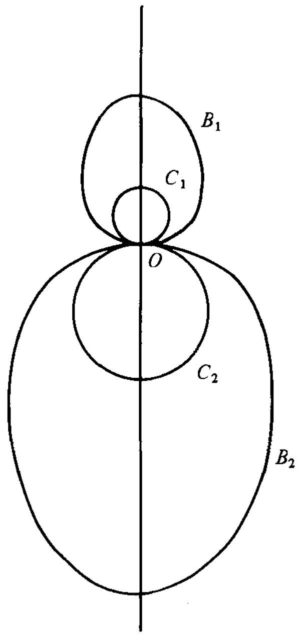

FIGURE 3. Two touching axisymmetrical bodies.

where the sign of α is chosen to be positive or negative according as B_2 touches B_1 internally or externally, it is clear that z/r^2 is a similarity variable and it is natural to introduce coordinates ξ, η defined by

$$\xi = \frac{2z}{r^2}, \quad \eta = \frac{2}{r}. \tag{3.3}$$

The surfaces ξ = constant are paraboloids which touch at O and η = constant are cylinders with axes along the axis of symmetry of the bodies. Although the level surfaces are not orthogonal in general, they approach orthogonality as $\eta \to \infty$, i.e. as the point of contact between B_1 and B_2 is approached. Near O the bodies are defined by $\xi = 1, \alpha$.

It is assumed that there is an axisymmetric steady or quasisteady Stokes flow about the bodies, thus the cylindrical polar components of velocity $(u, 0, w)$ can be expressed in terms of a stream function ψ as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{3.4}$$

where ψ satisfies the equation

$$\mathrm{L}_{-1}^2 \psi = \left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]^2 \psi = 0. \tag{3.5}$$

The boundary conditions of no slip on the bodies require that

$$\psi = \frac{\partial \psi}{\partial n} = 0. \tag{3.6}$$

In terms of the ξ, η variables, (3.6) are

$$\psi = \frac{\partial \psi}{\partial \xi} = 0 \quad (\xi = 1, \alpha),$$

and noting that as $\eta \to \infty$

$$\mathrm{L}_{-1}\{\eta^{-1}F(\xi,\eta)\} \sim \eta^3\left[\frac{\partial^2 F}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial F}{\partial \eta} + \frac{\partial^2 F}{\partial \xi^2}\right], \tag{3.7}$$

it follows that the solutions for the stream function near the point O can be represented by

$$\psi = \eta^{-\frac{3}{2}}\{(A+\xi C)\sinh s\xi + (B+\xi D)\cosh s\xi\}\,\mathrm{e}^{\pm is\eta}, \tag{3.8}$$

with the + or − sign chosen so that $\psi \to 0$ as $\eta \to \infty$ in order to satisfy the boundary conditions at the point of contact. Comparing (3.8) with (2.10), it is clear that s must satisfy the same equation (2.13) of §2, and consequently the stream function in the neighbourhood of O has the form

$$\psi = \eta^{-\frac{3}{2}}\,\mathrm{Re}\,A_1 f_1(\xi)\,\mathrm{e}^{-\lambda_1\eta/(1-\alpha)}, \tag{3.9}$$

unless $A_1 \neq 0$, when the appropriate asymptotic form is then

$$\psi = \eta^{-\frac{3}{2}}\,\mathrm{Re}\,\hat{A}_1 g_1(\xi)\,\mathrm{e}^{-\mu_1\eta/(1-\alpha)}, \tag{3.10}$$

with $f_1(\xi)$ and $g_1(\xi)$ given by (2.18) and (2.19). It therefore follows that the analysis and deductions for the two-dimensional flows discussed in §2 carry over completely to the axisymmetric three-dimensional problem. In general, the flow will not be purely antisymmetrical about the midsurface $\xi = \frac{1}{2}(\alpha+1)$, and (3.9) will be the correct asymptotic form in the neighbourhood of the point of contact.

The equations determining the branches of the stream function $\psi = 0$ which detach from the bodies and the separation points are again given by (2.20) and (2.21). The angle of separation will therefore in general tend to the limit 58.61° as the point of contact is approached. For a purely antisymmetrical flow, such as axisymmetric stagnation-point flow past two equal spheres, discussed by Davis (1979), only terms involving μ_n appear in the solution. The stream function is therefore given asymptotically by (3.10), which leads to a limiting separation angle given by (2.24), i.e. 48.15°. However, this example of antisymmetric flow exists only if the spheres have equal radii. For unequal-sized spheres the dominant term arises from the symmetric part of the flow, and this leads to a limiting separation angle of 58.61°.

4. Axisymmetrical bodies with conical cusps

Consider a body B with a re-entrant boundary which is locally a conical cusp. Examples of such a body are the closed torus or cardioid of revolution. If the body is symmetric about an axis through the cusp then in terms of cylindrical polar coordinates with pole at the vertex O of the cusp and z-axis along the axis of symmetry, the equation of the body can be written as

$$r = F(z) = \tfrac{1}{2}F''(0)\,z^2 + O(z^3) \tag{4.1}$$

in the neighbourhood of O. Letting r and z be dimensionless relative to $1/|F''(0)|$, assuming $F''(0) \neq 0$, the equation of the body near O has the form

$$r = \tfrac{1}{2}z^2 + O(z^3). \tag{4.2}$$

A suitable transformation of variables is now

$$\xi = \frac{2}{z}, \quad \eta = \frac{2r}{z^2}, \tag{4.3}$$

in which case the body B is given by $\eta = 1$ near O, where $\xi \gg 1$ and the axis of symmetry is given by $\eta = 0$.

If the flow in the neighbourhood of O is symmetric about $\eta = 0$, the stream function ψ satisfies the equation

$$L_{-1}^2 \psi = 0, \tag{4.4}$$

the operator being defined in (3.5). A solution of (4.4) is required that satisfies the boundary conditions

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \quad (\eta = 1) \tag{4.5}$$

and $\psi \to 0$ as $\xi \to \infty$.

For large ξ, the level surfaces ξ = constant and η = constant are approximately orthogonal, and the relation

$$L_{-1}\{\xi F(\xi, \eta)\} \sim \xi^3 \left\{\frac{\partial^2 F}{\partial \xi^2} - \frac{1}{\eta}\frac{\partial F}{\partial \eta} + \frac{\partial^2 F}{\partial \eta^2}\right\}$$

holds when $\xi \gg 1$. Thus a solution of (4.4) which is bounded on $\eta = 0$ is

$$\psi = \xi^{-3} \eta \{A I_1(s\eta) + B\eta I_0(s\eta)\} \mathrm{e}^{\pm \mathrm{i} s \xi}, \tag{4.6}$$

where I_n is the modified Bessel function of the first kind of order n ($n = 0, 1$). In (4.6) the sign is chosen to ensure that $\psi \to 0$ as $\xi \to \infty$ and the real part of the expression for ψ is understood to be taken. The boundary conditions on the body lead to the homogeneous equations

$$A I_1(s) + B I_0(s) = 0, \tag{4.7}$$

$$A[I_1(s) + s I_1'(s)] + B[2 I_0(s) + s I_0'(s)] = 0, \tag{4.8}$$

and consistency of these equations in turn leads to the equation

$$[I_1(s)]^2 - I_0(s) I_2(s) = 0. \tag{4.9}$$

Excluding $s = 0$, which leads to the trivial solution for ψ, the solutions of (4.9) are $s = \pm \zeta_n$ and their conjugates, with ζ_n lying in the first quadrant of the complex s-plane, and arranged in order of increasing imaginary part. The first two values of ζ_n were calculated by Dorrepaal *et al.* (1976) and found to be

$$\zeta_1 = 1.467 + 4.466\mathrm{i}, \quad \zeta_2 = 1.727 + 7.694\mathrm{i}. \tag{4.10}$$

A property of the solutions of (4.9) is that

$$\operatorname{Re}(\zeta_n) > \operatorname{Re}(\zeta_{n-1}), \quad \operatorname{Im}(\zeta_n) > \operatorname{Im}(\zeta_{n-1}) \quad (n \geqslant 2)$$

and that $\operatorname{Im}(\zeta_n)$ increases rapidly with n, so other values of ζ_n can be easily calculated using Newton's method and the asymptotic expansions of the Bessel functions for large $|\zeta|$. The analogous equation

$$J_1(\beta_1)^2 - J_0(\beta) J_2(\beta) = 0$$

arises in Sonshine, Cox & Brenner's (1965) analysis of the Stokes translation of a particle of arbitrary shape along the axis of a circular cylinder. These authors have calculated the first 46 values of β correct to eight significant figures. Hence, noting that $\zeta_n = \mathrm{i}\bar{\beta}_n$, further solutions for ζ_n ($n > 2$) are available, but will not be required here.

It therefore follows that near the cusp the stream function has the asymptotic form

$$\psi = \xi^{-3} \eta \operatorname{Re} A_1 \left\{I_1(\zeta_1 \eta) - \frac{\eta I_0(\zeta_1 \eta) I_1(\zeta_1)}{I_0(\zeta_1)}\right\} \mathrm{e}^{\mathrm{i}\zeta_1 \xi}, \tag{4.11}$$

where $\xi \gg 1$. For any fixed value of η, the stream function vanishes infinitely many times as the vertex of the cusp is approached, indicating the existence of a nested set of toroidal vortices whose centres lie along the z-axis. The equations of the branches $\psi = 0$ which span the walls of the cusp and form the boundaries of the toroidal vortices are given by

$$-\xi \operatorname{Re}(\zeta_1) = \arg A_1 + \arg f_1(\eta) + (m - \tfrac{1}{2})\pi \quad (\eta \neq 0, 1), \tag{4.12}$$

where

$$f_1(\eta) = I_1(\zeta_1 \eta) - \frac{\eta I_0(\zeta_1 \eta) I_1(\zeta_1)}{I_0(\zeta_1)}.$$

These surfaces meet the z-axis or $\eta = 0$ in stagnation points at which $\xi = \xi_M$, where

$$-\xi_M \operatorname{Re}(\zeta_1) = \arg A_1 + \arg\left[\zeta_1 - \frac{2I_1(\zeta_1)}{I_0(\zeta_1)}\right] + (m - \tfrac{1}{2})\pi. \tag{4.13}$$

Separation from the boundary of the body occurs where $\xi = \xi_s$, $\eta = 1$, with

$$-\xi_s \operatorname{Re}(\zeta_1) = \arg A_1 + \arg I_2(\zeta_1) + (m - \tfrac{1}{2})\pi. \tag{4.14}$$

The angle of separation γ at a separation point is now

$$\gamma = \tan^{-1}\left[3 \frac{\partial^3 \psi}{\partial \xi\, \partial \eta^2} \Big/ \frac{\partial^3 \psi}{\partial \eta^3}\right],$$

which can be shown to reduce to

$$\gamma = \tan^{-1}\left\{\frac{3 \operatorname{Re}(\zeta_1)}{\operatorname{Im}[\zeta_1 I_0(\zeta_1)/I_1(\zeta_1)]}\right\} = 45.25^\circ. \tag{4.15}$$

Thus, as the vertex of the cusp is approached, the angle at which the flow separates from the boundary of the body tends to this angle. This is a result which is generally true (i.e. $A_1 \neq 0$) for any axisymmetrical Stokes flow close to a cylindrical cusp. A particular body having this geometrical feature is the closed torus, and the separation angle of 45.25° may be verified directly in this case using the global solution for streaming flow past this body which was obtained by Dorrepaal *et al.* (1976).

5. Flow between parallel planes

An arbitrary Stokes flow between parallel planes $y = 0$ and $y = 1$ is considered. The flow is such that the non-slip condition on the planes is satisfied and a condition consistent with the decay of flow as $x \to \infty$, say. The purpose of this section is to examine the structure of the flow as $x \to \infty$. The problem is indicated schematically in figure 4.

A solution for the biharmonic stream function satisfying the boundary conditions is

$$\psi = \{(A + By)\sinh sy + C\cosh sy\}\,e^{\pm isx}, \tag{5.1}$$

with the sign of s chosen to ensure $\psi \to 0$ as $x \to +\infty$. To satisfy the conditions on the planes leads to a set of homogeneous equations for A, B and C, and the consistency of these equations requires that

$$\sinh^2 s = s^2. \tag{5.2}$$

The roots of this equation are $s = \pm i\lambda_n$, $\pm\mu_n$ and their conjugates, with λ_n, μ_n given in §2. Thus the form taken by the stream function when $x \gg 1$ is generally given by

$$\psi = \operatorname{Re}\{A_1\{(1-y)\sin\lambda_1 y + y\sin\lambda_1(1-y)\}\,e^{-\lambda_1 x}\}, \tag{5.3}$$

except when $A_1 = 0$, in which case the appropriate asymptotic form is then

$$\psi = \operatorname{Re}\{\hat{A}_1\{(1-y)\sin\mu_1 y - y\sin\mu_1(1-y)\}\,e^{-\mu_1 x}\}. \tag{5.4}$$

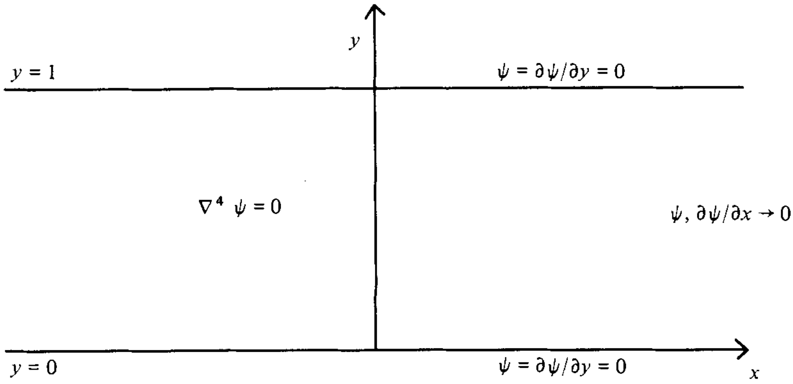

FIGURE 4. Flow between two parallel planes.

These are the forms of solution which result in corner flow between planes which intersect at angle α when $\alpha \to 0$ is such a way that a fixed distance between a fixed pair of points, one on either plane, is maintained, as discussed by Moffatt (1964). In that context (5.3) provides the limiting form corresponding to antisymmetric flow and (5.4) provides the limiting form to symmetric flow. It is therefore apparent that an arbitrary Stokes flow of the type postulated eventually becomes antisymmetric as $x \to \infty$. Furthermore, the flow pattern is such that an infinite set of eddies in a cellular formation develops. The close similarity between (5.3), (5.4) and (2.16), (2.17) allows one to infer that the equations of the eddy cell boundaries where $\psi = 0$ are

$$x \operatorname{Im}(\lambda_1) = \arg A_1 + \arg\{(1-y)\sin\lambda_1 y + y\sin\lambda_1(1-y)\} + (m-\tfrac{1}{2})\pi \quad (y \neq 0, 1). \quad (5.5)$$

The cells therefore have constant length $\pi/\operatorname{Im}(\lambda_1) = 1.396$, as well as height. The points of separation from the planes where these curves intersect $y = 0$ and $y = 1$ occur where $\partial^2\psi/\partial y^2 = 0$, and the solutions $x = x_s$ are given by

$$x_s \operatorname{Im}(\lambda_1) = \arg A_1 + \arg\lambda_1 + \arg(1+\cos\lambda_1) + (m-\tfrac{1}{2})\pi, \quad (5.6)$$

with $y = 0$ or 1.

The angle of separation at a typical separation point $x = x_s$, $y = 1$ is evidently

$$\gamma = \tan^{-1}\left[3\frac{\partial^3\psi}{\partial x\,\partial y^2}\Big/\frac{\partial^3\psi}{\partial y^3}\right], \quad (5.7)$$

and by comparison with the analysis of §2 it is at once clear that $\gamma \to 58.61°$ as $x \to \infty$. This is the limiting angle of separation of any Stokes flow of the type considered, provided that its leading term for large positive values of x is given by the antisymmetric asymptotic form (5.3). An example is the flow produced by rotating a circular cylinder between two parallel planes. Here the flow would decay to zero at $x = -\infty$ as well as at $x = +\infty$, which would imply that a cellular eddy structure is formed as $x \to -\infty$ also. The separation points would appear on each of the planes when $x = \pm x_s$, with x_s given asymptotically by (5.6). The angles of separation would now be γ when $x = x_s$ and $\pi - \gamma$ when $x = -x_s$. For this type of flow the stream function would be an even function of x, and if the cylinder were placed symmetrically between the planes, the stream function would also be an even function of $y - \frac{1}{2}$. Thus only the eigenvalues μ_n would contribute to the solution. However, if the cylinder were not centrally located between the planes, the stream function would then no longer be an even function of $y - \frac{1}{2}$ for all x but would become such a function as

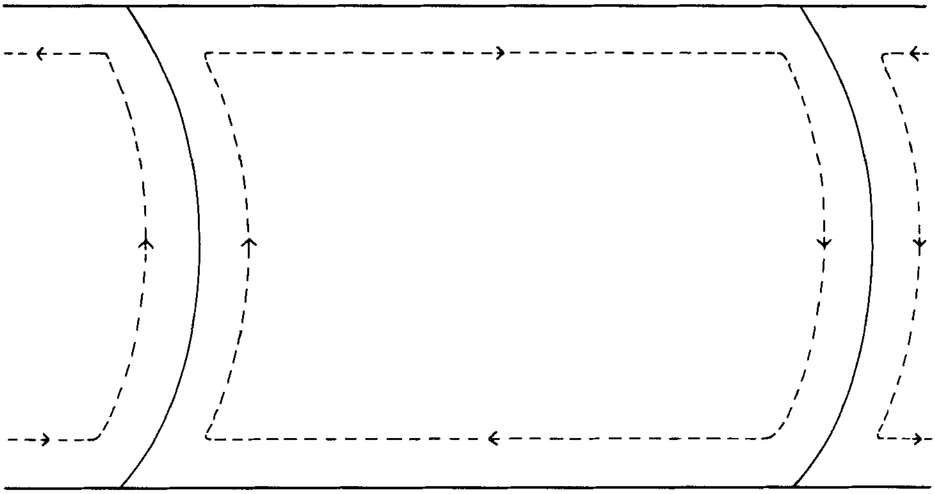

FIGURE 5. The separation streamlines for flow between parallel planes. The broken-line curves indicate the general direction of flow within the cells.

$|x| \to \infty$, resulting in a limiting separation angle of 58.61°. This particular angle of separation is somewhat surprising in view of the symmetrical character of both the geometry and streamline pattern on either side of the cylinder.

A recent global study of a flow between parallel planes has been carried out by W. Hackborn and K. B. Ranger (1982 private communication), in which a line rotlet is placed at any position between two parallel planes. An exact solution is found for the stream function for the flow, and the asymptotic structure of this solution at large $|x|$ is entirely in accord with (5.3) and verifies the predicted value of 58.61° for the limiting angle of separation.

From (5.5), the values x_M at which the cell boundaries $\psi = 0$ cross the midplane $y = \frac{1}{2}$ are given by

$$x_M \operatorname{Im}(\lambda_1) = \arg A_1 + \arg(\sin\tfrac{1}{2}\lambda_1) + (m - \tfrac{1}{2})\pi. \tag{5.8}$$

The distance $x_M - x_s$ gives the maximum displacement of a separation curve from the common perpendicular to the two planes through a separation point. This has the constant value 0.1451. It is interesting to note that this value produced by an asymptotic analysis for large $|x|$ is reproduced in Hackborn and Rangers' exact analysis from the second separation curve for all positions of the rotlet. Indeed, from the second cell not enclosing the rotlet and thereafter, the streamline pattern is that of antisymmetric flow, again no matter where the rotlet is located between the planes. This shows that the parallel planes plus the condition of evanescence quickly establishes the character of the flow whatever the precise type of flow generator. Figure 5 shows the separation streamlines which form the cell boundaries.

For purely symmetric flow, when the leading term in ψ as $x \to +\infty$ is given by (5.4), a comparison with the analysis of §2 shows that the limiting angle of separation is now

$$\tan^{-1}\left[-\frac{3\operatorname{Im}(\mu_1)}{\operatorname{Im}(\cos\mu_1)}\right] = 48.15°. \tag{5.9}$$

Hackborn and Ranger have also considered the flow generated between two planes by two line rotlets of opposite sense but equal strength placed symmetrically between and on a line perpendicular to both planes. This gives an example of a flow which is purely symmetric about $y = \frac{1}{2}$. The results of these authors corroborate both the flow pattern and separation angle given by (5.9).

The limiting angles of separation of 58.61° for antisymmetric flow and 48.15° for symmetric flow between parallel planes have also been established by Hancock (1983). He has calculated the angle of separation from Moffatt's solution for flow between planes which intersect at an angle of α. Then, by letting $\alpha \to 0$ in such a way that the distance between two fixed points, one on either plane, remains constant, the appropriate angles for symmetric and antisymmetric flow between parallel planes are retrieved. The nature of Hancock's limiting process is such that the vertex of the wedge-shaped fluid region moves to infinity. This ensures that the asymptotic behaviour of the stream function as infinity is approached in at least one direction is exactly that prescribed in this section.

Another related result concerns axisymmetric flow between parallel planes when the axis of symmetry is perpendicular to both planes. In terms of cylindrical polar coordinates (r, θ, z) in which the planes are given by $z = 0, 1$, the boundary conditions require

$$\psi = \frac{\partial \psi}{\partial z} = 0 \quad (z = 0, 1), \tag{5.10}$$

$$\psi, \frac{\partial \psi}{\partial r} \to 0 \quad (r \to \infty,\ 0 < z < 1), \tag{5.11}$$

where ψ is the axisymmetric stream function. The leading term of the asymptotic expansion for ψ when $r \gg 1$ is in general given by

$$\begin{aligned}\psi &= \operatorname{Re}\{A_1\{(1-z)\sin\lambda_1 z + z\sin\lambda_1(1-z)\}\, rK_1(\lambda_1 r)\}\\ &\sim \operatorname{Re}\left\{A_1\{(1-z)\sin\lambda_1 z + z\sin\lambda_1(1-z)\}\left(\frac{\pi r}{2\lambda_1}\right)^{\frac{1}{2}} \mathrm{e}^{-\lambda_1 r}\right\},\end{aligned} \tag{5.12}$$

with K_1 the modified Bessel function of the second kind of order unity and $A_1 (\neq 0)$ is an undetermined constant. It therefore follows from analyses given elsewhere in this paper that for large r the flow is antisymmetric about the plane $z = \frac{1}{2}$ and composed of a set of nested toroidal vortices which detach from the planes at angles of 58.61°. An example of this flow would be the quasisteady flow caused by the translation of a small sphere along a direction perpendicular to both planes. If $A_1 = 0$, such as in purely symmetrical flow about $z = \frac{1}{2}$, the leading term in the asymptotic expansion for ψ is then

$$\psi = \operatorname{Re}\left\{\hat{A}_1\{(1-z)\sin\mu_1 z - z\sin\mu_1(1-z)\}\left(\frac{\pi r}{2\mu_1}\right)^{\frac{1}{2}} \mathrm{e}^{-\mu_1 r}\right\}. \tag{5.13}$$

Now there is a system of double vortices above and below the plane $z = \frac{1}{2}$, and the angles of separation are now 48.15°.

6. Flow in a circular cylinder

A circular cylinder of radius unity contains viscous fluid whose motion may be described as a steady or quasisteady Stokes flow which is symmetric about the axis of the cylinder. The flow may be generated in an arbitrary manner but is subject to the constraints of the no-slip boundary condition on the wall and that the motion decays to zero as $z \to +\infty$, where (r, θ, z) are cylindrical polar coordinates with z-axis along the axis of the cylinder. The velocity components can be expressed in terms of a stream function ψ by

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{6.1}$$

where

$$\mathrm{L}_{-1}^2\psi = \left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]^2 \psi = 0. \tag{6.2}$$

The solution of (6.2) in separable variables has the form

$$\psi = r\,\mathrm{Re}\{AI_1(sr) + BrI_0(sr)\}\,\mathrm{e}^{\pm isz}, \tag{6.3}$$

with the sign chosen so that $\psi \to 0$ as $z \to +\infty$. Satisfaction of the no-slip condition leads to the following consistency condition:

$$[I_1(s)]^2 - I_0(s)\,I_2(s) = 0. \tag{6.4}$$

Equation (6.4) is exactly the same eigenvalue equation as (4.9), and it therefore follows that for $z \gg 1$ the stream function has the form

$$\psi = r\,\mathrm{Re}\,A_1\left\{I_1(\zeta_1 r) - \frac{rI_0(\zeta_1 r)\,I_1(\zeta_1)}{I_0(\zeta_1)}\right\}\mathrm{e}^{i\zeta_1 z}, \tag{6.5}$$

assuming that $A_1 \neq 0$. Consequently, for large positive z the flow has a cellular toroidal vortex structure. The branches of $\psi = 0$ which span the cylinder are the boundaries of the cells, and their equations are given asymptotically by (4.12) with r replacing η. Likewise, the points of separation on the cylinder are given by $r = 1, z = z_s$, with z_s determined from (4.14) when ξ_s is replaced by z_s, and the positions of the stagnation points on the axis can be found by replacing ξ_M by z_M in (4.13). The angle γ at which the flow separates from the cylinder is given by (4.15). Thus, for $z \gg 1$, the flow separates from the wall of the cylinder at an angle of 45.25°. The distance $z_M - z_s$ measures the maximum displacement of the separation curves from the radii through separation points. It is a constant and is given by

$$\begin{aligned} z_M - z_s &= \{\pi + \arg\zeta_1 - \arg[\zeta_1 I_0(\zeta_1) - 2I_1(\zeta_1)]\}/\mathrm{Re}\,(\zeta_1) \\ &= 0.428. \end{aligned} \tag{6.6}$$

The length of each cell is also constant, and its value is $\pi/\mathrm{Re}\,(\zeta_1) = 1.599$. Figure 6 shows the traces of the separation stream surfaces in any plane through the axis of the cylinder. These form the boundaries of the toroidal eddy cells.

An obvious example of a flow satisfying the constraints of this problem is the slow motion of a sphere along the axis of a circular cylinder. In this case the flow decays when $z \to -\infty$ as well as when $z \to +\infty$, so infinite systems of closed eddies form both ahead of and behind the sphere at sufficiently large distances along the cylinder.

7. Discussion

It may be somewhat unexpected that the same angles of separation should occur in such physically dissimilar problems as flow near a line or point of contact between two bodies and in the far flow field between parallel planes or flow near the vertex of a cusp and the far flow field in a circular cylinder. However, it is a simple matter to establish that the two types of problems are mathematically equivalent.

Consider first a two-dimensional Stokes flow in the (x, y)-plane. If the coordinates are inverted with respect to the unit circle with centre at the origin, a point $P(x, y)$

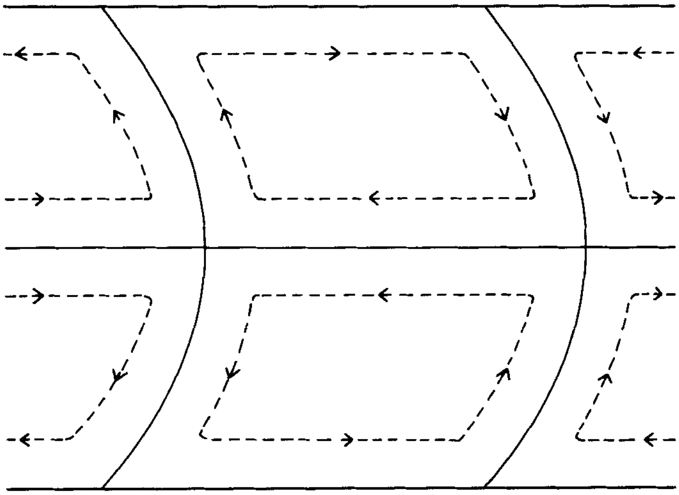

FIGURE 6. The separation streamlines for flow in a circular cylinder. The broken-line curves indicate the general direction of flow within the cells.

inverts into a point $P'(x', y')$ with

$$x' = \frac{x}{x^2+y^2}, \quad y' = \frac{y}{x^2+y^2}. \tag{7.1}$$

It therefore follows that if ψ satisfies the plane biharmonic equation

$$\nabla^4\psi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = 0 \tag{7.2}$$

then $\psi' = \psi/(x^2+y^2)$ satisfies

$$\nabla'^4\psi' = \left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)^2 \psi' = 0. \tag{7.3}$$

Noting that $x'^2+y'^2 = (x^2+y^2)^{-1}$, it follows that

$$x = \frac{x'}{x'^2+y'^2}, \quad y = \frac{y'}{x'^2+y'^2}, \quad \psi = \frac{\psi'}{x'^2+y'^2}. \tag{7.4}$$

Thus the relations connecting x, y, ψ and x', y', ψ' are reciprocal. If, however, $x' \gg 1, y' = O(1)$ then

$$x \sim \frac{1}{x'}, \quad y \sim \frac{y'}{x'^2},$$

giving

$$y' \sim \frac{y}{x^2}, \quad x' \sim \frac{1}{x},$$

and identifying x' with $\frac{1}{2}\eta$ and y' with $\frac{1}{2}\xi$, (2.6) are recovered. Thus a Stokes flow in the neighbourhood of the origin of the (x, y)-plane in the region between the curves $y = \frac{1}{2}x^2$ and $y = \frac{1}{2}\alpha x^2$ transforms into a Stokes flow in the region of the (ξ, η)-plane between $\xi = 1$ and $\xi = \alpha$ for which $\eta \gg 1$. The stream functions $\psi(x, y)$ and $\psi'(\xi, \eta)$ for the two flows are related by the equation

$$\lim_{x,y\to 0} \psi(x, y) = 4 \lim_{\eta\to\infty} \eta^{-2}\psi'(\xi, \eta). \tag{7.5}$$

Consequently if separation occurs in one flow it must also occur in the other flow and the limiting angles of separation as $x, y \to 0$ and $\eta \to \infty$ must be identical.

For axisymmetric motion, the point $P(r, \theta, z)$ is inverted into the point $P'(r', \theta, z')$ with respect to a unit circle in a meridional plane with centre at the origin. Thus

$$r' = \frac{r}{r^2+z^2}, \quad z' = \frac{z}{r^2+z^2}. \tag{7.6}$$

Now it can be shown that if $L_{-1}^2 \psi = 0$ then

$$L_{-1}'^2 \left\{ \frac{\psi}{(r^2+z^2)^{\frac{3}{2}}} \right\} = 0,$$

where the operator is defined in (6.2) and the prime indicates that primed variables replace unprimed variables. If now r' and z' are identified with $\frac{1}{2}\eta$ and $\frac{1}{2}\xi$ respectively, then the relations (3.3) are recovered when $\xi = O(1), \eta \gg 1$. This means that the region near the origin $r = z = 0$ bounded by the surfaces $z = \frac{1}{2}r^2$ and $z = \frac{1}{2}\alpha r^2$ is mapped into the region between the planes $\xi = 1, \alpha$ for which $\eta \gg 1$. The relation connecting the stream functions for flow in the two regions is

$$\lim_{r,z \to 0} \psi(r, \theta, z) = 8 \lim_{\eta \to \infty} \eta^{-3} \psi'(\xi, \theta, \eta), \tag{7.7}$$

which is confirmed on comparing (3.9) and (3.10) with (5.12) and (5.13) respectively. If, however, $\xi \gg 1$, $\eta = O(1)$, (4.3) are recovered, and the equivalence of Stokes flow near the vertex of a conical cusp and the flow in a circular cylinder at a great distance from the generating disturbance is established. The relation between the stream functions is now

$$\lim_{r,z \to 0} \psi(r, \theta, z) = 8 \lim_{\xi \to \infty} \xi^{-3} \psi'(\xi, \theta, \eta), \tag{7.8}$$

which is confirmed on comparing (4.11) with (6.5). Again separation in one flow implies separation in the other flow with identical limiting angles of separation when $r, z \to 0$ and $\xi \to \infty$.

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